FROBENIUS LIFTABILITY AND LIPMAN-ZARISKI CONJECTURE

KENTA SATO

This talk is based on joint work with Tatsuro Kawakami ([KS25]).

1.1. **Frobenius liftability.** Let k be a field of characteristic $p > 0$ and $W_2(k)$ be the ring of Witt vectors of length 2. For example, we have $W_2(\mathbb{F}_p) \cong \mathbb{Z}/p^2\mathbb{Z}$.

Definition 1. We say that a k-algebra R is *F-liftable* if there exist

- a flat $W_2(k)$-algebra $\widetilde{R}$ with $\widetilde{R} \otimes_{W_2(k)} k \cong R$, and
- an $\widetilde{R}$-algebra homomorphism $\widetilde{F}\colon \widetilde{R} \longrightarrow \widetilde{R}$ which fits into the following commutative diagram:

$$\begin{array}{ccc} \widetilde{R} & \xrightarrow{\widetilde{F}} & \widetilde{R} \\ \downarrow & & \downarrow \\ R & \xrightarrow{F} & R \end{array}.$$

For example, every (structure ring of) toric variety is F-liftable. On the other hand, it has become evident that F-liftability imposes strong conditions on singularities (see [KW24, Theorem B] for example).

1.2. **Lipman–Zariski conjecture.** Let $(R, \mathfrak{m})$ be a normal local domain essentially of finite type over $k = \overline{k}$. A *tangent module* of R is the dual

$$T_R \coloneqq \mathrm{Hom}_R(\Omega_{R/k}, R).$$

of the module of Kähler differentials $\Omega_{R/k}$. In characteristic zero, it is conjectured that singularities with free tangent modules are smooth. This is known as the Lipman–Zariski Conjecture [Lip65]. The conjecture has been confirmed in numerous cases, and specifically for log canonical singularities, it was proved in [Dru14, GK14].

In contrast, in characteristic $p > 0$, Lipman [Lip65] observed that rational double points (RDPs) of type A_n violate the conjecture when $n+1$ is divisible by p. Notably, these singularities are all strongly F-regular, and even more specifically, they are toric. As such, achieving smoothness appears unattainable in characteristic $p > 0$, even under strong assumptions beyond the freeness of the tangent modules. Instead, we prove the following result: strongly F-regular singularities with free tangent modules are F-liftable rather than smooth.

Theorem 2. *Let $(R, \mathfrak{m})$ be a strongly F-regular local domain essentially of finite type over a perfect field of characteristic $p > 0$. If T_R is free, then R is F-liftable.*

References

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Department of Mathematics and Informatics, Chiba university, Chiba, 263-8522, Japan

Email address: sato@math.s.chiba-u.ac.jp